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Origin of Logarithmic Operators in Conformal Field Theories

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Abstract

We study logarithmic operators in Coulomb gas models, and show that they occur when the "puncture" operator of the Liouville theory is included in the model. We also consider WZNW models for $SL(2,\mathcal{R})$, and for $SU(2)$ at level 0, in which we find logarithmic operators which form Jordan blocks for the current as well as the Virasoro algebra.

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1 Introduction

Logarithmic operators in conformal field theory were first studied by Gurarie in the $c = -2$ model [1]. These logarithms have now been found in a multitude of other models such as the WZNW model on the supergroup GL(1,1) [2], the gravitationally dressed CFTs [3], $c_{p,1}$ and non-minimal $c_{p,q}$ models [1, 4, 5, 6], critical disordered models [7, 8], and WZNW models at level 0 [9, 10], and play a role in the study of critical polymers and percolation [4, 5, 11, 12], $2D$-magneto-hydrodynamic turbulence and ordinary turbulence [13, 14] and quantum Hall states [15, 16]. They are also important for studying the problem of recoil in the theory of strings and D-branes [9, 17, 18, 19, 20] as well as target-space symmetries in string theory in general [9]. The representation theory of the Virasoro algebra for logarithmic CFT was developed in [21].

In this paper we discuss the free field formulation of CFT with logarithmic operators, and we show that they are closely related to the "puncture" operator of 2D gravity. In sections 2, 3 and 4, we discuss $c_{p,q}$ models, and gravitationally dressed CFT. In section 5, we consider the analogous situation in models with an affine Lie algebra as well as a Virasoro algebra. It was first realised in [1] that when there are logarithmic operators, the generators of the Virasoro algebra cannot be diagonalized, but have a Jordan cell structure:

$$\begin{aligned} L_0|C\rangle &= h|C\rangle \\ L_0|D\rangle &= h|D\rangle + |C\rangle \end{aligned} \qquad (1)$$

There are two possible types of Jordan cell structures that we can consider in the analogous case when there is a current* as well as a Virasoro algebra. The first possibility, which is directly analogous to eq. (1), is to have operators with the behaviour (first introduced in [23]):

$$J_0^a C = t^a C$$

*The affine Lie algebra (current algebra) is often referred to as the Kac-Moody algebra, althogh this may be misleading as the affine algebras were discovered independently of the Kac-Moody algebra, and in the correct mathematical nomenclature "Kac-Moody algebra" refers to a more general case. For a short history of the subject see the appendix of [22].

$$
\begin{aligned}
J_0^a D &= t^a D + C \\
J_n^a C = J_n^a D &= 0, \qquad\qquad n \geq 1.
\end{aligned} \tag{2}
$$

Perhaps surprisingly, the operators C and D with this behaviour do not also form Jordan cells for the Virasoro algebra of the WZNW model (although they might in other models), so although they have the same conformal dimensions, they are both primary, not logarithmic, operators of the Virasoro algebra. The second possibility is to have a pair of operators which form a Jordan cell for L_0 but not for J_0^a:

$$
\begin{array}{llll}
 & J_0^a C = t^a C, & J_n^a C = 0, & n \geq 1 \\
J_0^a D = t^a D, & J_1^a D = t^a K, & J_n^a D = 0, & n \geq 2 \\
 & J_0^a K = t^a K, & J_n^a K = 0, & n \geq 1.
\end{array} \tag{3}
$$

In this case, C and D are a logarithmic pair, obeying eq. (1), while K is a primary field with a dimension 1 lower than that of C and D. It is D and K that form a Jordan cell for J_1^a. Similar constructions are possible where the dimensions differ by integers greater than 1.

Let us begin by reviewing how logarithmic operators appear in conformal field theory. Let us consider for example the four-point correlation functions of a primary field $\mu(z)$ with anomalous dimensions h, such that $< \mu(z)\mu(0) >= z^{-2h}$. This correlation function can be represented as

$$
< \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4) >= \frac{1}{|(z_1-z_2)(z_3-z_4)|^{4h}} \sum_{i,j} U_{ij} F^i(x) F^j(\bar{x}), \tag{4}
$$

where $x = (z_1-z_2)(z_3-z_4)/(z_1-z_4)(z_3-z_2)$. In many different models the unknown functions $F^i(x)$ (the conformal blocks) are given as solutions of the hypergeometric equation (or a higher order Fuchsian differential equation).

$$
x(1-x)\frac{d^2\mathcal{F}}{dx^2} + [d-(a+b+1)x]\frac{d\mathcal{F}}{dx} - ab\mathcal{F} = 0 \tag{5}
$$

which in general has two independent solutions

$$
\mathcal{F}_1 = F(a,b,d;x), \qquad \mathcal{F}_2 = x^{1-d} F(a-d+1, b-d+1, 2-d; x) \tag{6}
$$

where $F(a,b,d;x)$ is a hypergeometric function (we use d as the third parameter to avoid confusion with the central charge c) and these two independent solutions correspond to the two primary fields in the OPE of

$$\mu(z)\mu(0) = \frac{1}{z^{2h}} \left[I + z^{d-1} O + \cdots \right] \tag{7}$$

one of which is an identity operator I and the second operator O has an anomalous dimension $1-d$. It is simplest to discuss the case when there are only two primary fields in the OPE, although in general there are more; for example, in minimal models, the fields appearing in the OPE are given by

$$\phi_{r_1,s_1} \times \phi_{r_2,s_2} = \sum_{r_3=|r_1-r_2|+1}^{r_1+r_2-1} \sum_{s_3=|s_1-s_2|+1}^{s_1+s_2-1} \phi_{r_3,s_3} \tag{8}$$

If μ in eqs. (4) and (7) is the $(1,2)$ field in a minimal model, O is the $(1,3)$ field and the conformal blocks are given by eq. (5) with

$$\begin{aligned} a &= \frac{1}{24}\left[\sqrt{1-c} - \sqrt{25-c}\right]^2 \\ b &= -1 + \frac{1}{8}\left[\sqrt{1-c} - \sqrt{25-c}\right]^2 \\ d &= \frac{1}{12}\left[\sqrt{1-c} - \sqrt{25-c}\right]^2 \end{aligned} \tag{9}$$

In this way we recover the conformal blocks in a generic CFT [24]. However, this is not true if the parameter d in the hypergeometric equation (5) is an integer. There is a general theorem in the theory of the second order differential equations which deals with the expansion of the solution near the regular point $x=0$

$$x^\alpha \sum_n a_n x^n \tag{10}$$

This theorem tells us how to calculate the coefficients a_n if one knows the two roots α_1 and α_2, which are the solutions of the so-called indicial equation [25]. However, if the difference $\alpha_1 - \alpha_2$ is an integer, the second solution either equals the first one (when $\alpha_1 = \alpha_2$) or some of the coefficients are undefined. In both cases the second solution has logarithmic terms $x^n \ln x$ in the expansion (10), besides the usual terms x^n. For the hypergeometric equation the indicial equation is

$$\alpha(\alpha - 1 + d) = 0 \tag{11}$$

and the two roots are $\alpha_1 = 0$ and $\alpha_2 = d-1$, i.e. for integer $d = 1+m$ the second solution has logarithmic terms.

Note that the dimensions of primary fields that appear in the OPE (7) are given by the roots of the indicial equation. This can be seen by substituting the general OPE

$$\mu(z)\mu(w) = \frac{1}{(z-w)^{2h}} \sum \left\{ (z-w)^{h_i} O_i(w) + \cdots \right\}, \tag{12}$$

where h_i is the dimension of O_i, into the four-point function, giving (omitting the $\bar{z}$ dependence)

$$< \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4) >= \frac{1}{(z_1-z_2)^{2h}(z_3-z_4)^{2h}} \sum_i \left(\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_2-z_3)} \right)^{h_i} + \cdots. \tag{13}$$

Comparing this with eq. (4), we see that:

$$F_i(x) = x_i^h(1+\ldots) \tag{14}$$

So that h_i is given by α in eq. (10), which is a root of the indicial equation.

Therefore the logarithmic solutions occur when the dimension of the second operator O in the OPE (7) is degenerate either with the identity operator (when $m = 0$ and $d = 1$) or with one of its Virasoro descendants (for negative integer m when O has a positive dimension $|m|$). For positive integer m (in which case O itself has a negative dimension $-m$) one of its descendants will be degenerate with I (in both cases, the descendant in question is a null vector of the Virasoro algebra, as is discussed in section 4). If we had considered a more general four-point function containing two distinct primary fields, we would have a similar situation with another primary operator taking the place of the identity. If we had a higher order differential equation instead of the hypergeometric equation, the above discussion would apply whenever two of the roots of the indicial equation differed by an integer; in all cases, logarithmic conformal blocks occur when the dimensions of two of the operators in an OPE become degenerate.

The condition that $d = 1+m, \quad m \in Z$ is necessary, but not sufficient if $m \neq 0$. To have logarithms in this case one has to impose additional constraints on a and b [25],[26].

If $d = 1+m$, where m is a natural number, the two independent solutions are

$$\mathcal{F}_1 = F(a,b,1+m;x),$$
$$\mathcal{F}_2 = \log x \; F(a,b,1+m;x) + H(x), \tag{15}$$

where $H(x) = x^{-m} \sum_{k=0}^{\infty} h_k x^k$ and $h_m = 0$, unless either a or b equal $1 + m'$ with m' a natural number $m' < m$. In this case the second solution is only a polynomial in x^{-1} and the are no logarithms.

If $d = 1 - m$, where m is a natural number, the two independent solutions are

$$
\begin{aligned}
\mathcal{F}_1 &= x^m F(a+m, b+m, 1+m; x), \\
\mathcal{F}_2 &= \log x \; x^m F(a+m, b+m, 1+m; x) + H(x),
\end{aligned}
\tag{16}
$$

where $H(x)$ is again some regular expansion without logarithms, unless either a or b equal $-m'$ with an integer m' such that $0 \leq m' < m$, in which case $\mathcal{F}_1$ is a polynomial in x and there are no logarithms.

An example of a correlation function with logarithms is the four-point function of the $(1,2)$ operator in the model with $c = -2$ [1]. In this case, eq. (9) gives $a = b = 1/2$, $d = 1$, and the conformal blocks are given by eq. (15). The full correlation function, satisfying the constraints of crossing symmetry and locality, is:

$$
< \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4) >= |(z_1-z_3)(z_2-z_4)|^{-4h} |x(1-x)|^{-4h} \left[F(x)F(1-\bar{x}) + F(1-x)F(\bar{x}) \right]
\tag{17}
$$

where $h = -\frac{1}{8}$ and

$$
\begin{aligned}
F(x) &= F\left(\frac{1}{2}, \frac{1}{2}, 1; x\right) \\
F(1-x) &= \log x \; F\left(\frac{1}{2}, \frac{1}{2}, 1; x\right) + H(x)
\end{aligned}
\tag{18}
$$

As was shown in [1], this leads to the OPE

$$
\mu(z)\mu(0) = |z|^{1/2} \left[D(0) + \log |z|^2 C(0) + \cdots \right]
\tag{19}
$$

$D(z)$ is the logarithmic operator. The behaviour of the logarithmic pair C and D is expressed by the OPE's with the stress tensor:

$$
T(z)C(0) = \frac{h}{z^2} C(0) + \frac{1}{z} \partial C(0) + \cdots
\tag{20}
$$

$$
T(z)D(0) = \frac{h}{z^2} D(0) + \frac{1}{z^2} C(0) + \frac{1}{z} \partial D(0)
\tag{21}
$$

This implies that, instead of the usual irreducible representations, C and D and their descendants form an indecomposable representation of the Virasoro algebra, with C and D forming the basis of a Jordan cell for L_0 (as in eq. (1)).

2 Logarithmic and Pre-Logarithmic Operators in Coulomb Gas Models

We would like to understand the origin of logarithmic operators in models with the Liouville action, which describes 2D gravity in conformal gauge as well Coulomb gas models with $c < 1$:

$$S = \frac{1}{8\pi} \int d^2\xi \sqrt{g(\xi)} \left[\partial_\mu \phi(\xi) \partial^\mu \phi(\xi) + i\alpha_0 R^{(2)}(\xi)\phi(\xi) \right]. \tag{22}$$

The central charge of the above action is

$$c = 1 - 24{\alpha_0}^2 \tag{23}$$

and the stress tensor is:

$$T(z) = -\frac{1}{4} : \partial_z \phi \partial_z \phi : + i\alpha_0 \partial_z^2 \phi \tag{24}$$

The field ϕ has the short distance behaviour:

$$\phi(z)\phi(w) \sim -2 \log |z - w|^2 \tag{25}$$

In the case of gravitational dressing, ${\alpha_0}^2 < 1$, while for the (p, q) models, ${\alpha_0}^2 > 1$ and $c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$. In the case of a gravitationally dressed (p, q) model we have $c_L + c_{p,q} = 26$. In both models, the primary fields are vertex operators of the form

$$V_\alpha(z, \bar{z}) =: e^{i\alpha\phi(z,\bar{z})} : \tag{26}$$

with the conformal dimensions

$$h_\alpha = \alpha(\alpha - 2\alpha_0). \tag{27}$$

For the degenerate primary fields, h and α takes the values:

$$h_{r,s} = \frac{(rq - sp)^2 - (p - q)^2}{4pq}$$
$$\alpha = \alpha_{r,s} = \frac{1}{2}(1 - r)\alpha_+ + \frac{1}{2}(1 - s)\alpha_- \tag{28}$$

where,

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1} \tag{29}$$

The operators (26) have the usual OPE with the stress tensor:

$$T(z)V_\alpha(0) = \frac{h_\alpha}{z^2}V_\alpha(0) + \frac{1}{z}\partial V_\alpha(0) + \ldots \tag{30}$$

In minimal models, where $1 \leq r < q$ and $1 \leq s < p$, the exponential primary fields (26) and their descendants are the only fields in the theory. This changes if the primary field with the minimum dimension, $h_{0,0} = h_{q,p} = -\frac{(p-q)^2}{4pq}$, is included. This is particularly important for the $c_{p,1}$ models, in which the minimal model region ($1 \leq r < q$ and $1 \leq s < p$) is empty, and this field cannot be excluded. In these cases the models are known to contain logarithmic operators as well as ordinary primary fields [4, 6].

The exponential operator with the dimension $h_{0,0}$ has $\alpha = \alpha_0$. For every other value of α, there are two operators with the same dimension h_α given by eq. (27): V_α and $V_{(2\alpha_0-\alpha)}$. Since correlation functions can only be non-zero if they satisfy the condition $\sum \alpha = 2\alpha_0$, all correlation functions have to include one of the operators $V_{(2\alpha_0-\alpha)}$, with all the α's given by eq. (28) [27]. When $\alpha = \alpha_0$, there are also two primary operators with the same dimension; the second one is:

$$V_P = \left.\frac{\partial}{\partial\alpha}V_\alpha\right|_{\alpha=\alpha_0} = i\phi(z)e^{i\alpha_0\phi(z)}. \tag{31}$$

This is called the puncture operator in the Liouville theory (in which α_0 is imaginary). Since it contains ϕ and not just exponentials (or derivatives) of ϕ, we might expect it to have logarithmic correlation functions, but in fact V_P is the only operator of this form which is an ordinary primary operator, as can be seen by differentiating eq. (30) with respect to α, giving:

$$T(z)\left\{\frac{\partial}{\partial\alpha}V_\alpha(0)\right\} = \frac{\partial h_\alpha}{\partial\alpha}\frac{1}{z^2}V_\alpha(0) + \frac{h_\alpha}{z^2}\left\{\frac{\partial}{\partial\alpha}V_\alpha(0)\right\} + \frac{1}{z}\partial\left\{\frac{\partial}{\partial\alpha}V_\alpha(0)\right\} + \cdots \tag{32}$$

This is the same as the OPE for a primary field only when

$$\frac{\partial h_\alpha}{\partial\alpha} = 2(\alpha - \alpha_0) = 0, \tag{33}$$

The "puncture" operator, with the minimum dimension, is therefore an ordinary primary field, and has the usual 2- and 3-point functions with no logarithms. However, it turns out that there are logarithms in 4-point functions containing this operator. For example,

in the $c_{2,1} = -2$ model discussed in the previous section, eqs. (23) and (28) give $\alpha_0 = \alpha_{1,2} = 1/\sqrt{8}$, so the "puncture" operator is just the $(1,2)$ operator. We can understand why the logarithms appear in the four-point functions by observing that including the puncture operator in the model naturally leads to the inclusion of other operators of the form $i\phi e^{i\alpha\phi}$ (in this sense, the "puncture" operator could be called a "pre-logarithmic" operator), as can be seen by differentiating the OPE

$$e^{i\alpha\phi}(z,\bar{z})e^{i\beta\phi}(0) \sim \frac{e^{i(\alpha+\beta)\phi}(0)}{|z|^{2(h_\alpha+h_\beta-h_{\alpha+\beta})}} \tag{34}$$

giving:

$$i\phi e^{i\alpha_0\phi}(z,\bar{z})e^{i\beta\phi}(0) \sim \frac{i\phi e^{i(\alpha_0+\beta)\phi}(0)}{|z|^{2(h_{\alpha_0}+h_\beta-h_{\alpha+\beta})}} + 2\beta \log|z|^2 \frac{e^{i(\alpha_0+\beta)\phi}(0)}{|z|^{2(h_\alpha+h_\beta-h_{\alpha+\beta})}} \tag{35}$$

Eq. (35) has the form of the OPE (19) that arises in the limit where the dimensions of two operators become degenerate, leading to four point functions with logarithmic singularities. There is then a second operator, the logarithmic operator, for which instead of eq. (30) we find:

$$T(z)D_\alpha(0) = \frac{h_\alpha}{z^2}D_\alpha(0) + \frac{1}{z^2}V_\alpha(0) + \frac{1}{z}\partial D_\alpha(0) + \ldots \tag{36}$$

In the Coulomb gas picture, we can now see that the logarithmic operator D_α can be written as:

$$D_\alpha = \left(\frac{\partial h_\alpha}{\partial \alpha}\right)^{-1} \frac{\partial}{\partial \alpha} V_\alpha = \frac{i}{\alpha - \alpha_0}\phi e^{i\alpha\phi} \tag{37}$$

Note that this operator cannot be defined when $\alpha = \alpha_0$.

It is interesting to examine when the puncture operator, and therefore also logarithmic operators, will appear in a gravitationally dressed conformal field theory. In this case, the gravity sector is described by the action (22) with α_0 chosen to give the total central charge 26 [28, 29], so that if a model with central charge $c_{p,q}$ is coupled to gravity, we have

$$\alpha_0^2 = -1 - \frac{(p-q)^2}{4pq} \tag{38}$$

Since α_0 is imaginary, the vertex operators of the Liouville model are $V_\beta = e^{\beta\phi}$, with β real, and with dimensions [28]

$$h_\beta = -\beta(\beta - 2i\alpha_0) \tag{39}$$

The puncture operator, $V_P = \phi e^{i\alpha_0\phi}$, therefore has in this case the maximum rather than the minimum dimension

$$h_P = h_{i\alpha_0} = 1 + \frac{(p-q)^2}{4pq} \tag{40}$$

Primary fields $\Phi_{r,s}$ from the (p,q) model, with dimension $h_{r,s}$ are dressed by fields from the Liouville model with dimension h_β, to form composite fields with $\Phi_{r,s}e^{\beta\phi}$ with total conformal dimension 1, so that it makes sense to integrate the dressed fields over the surface [29]. We therefore have:

$$h_\beta + h_{r,s} = 1 \tag{41}$$

In a $c = 1$ model coupled to gravity, the puncture operator is a cosmological constant operator ($h_P = 1$), and logarithmic operators do exist in that model [3]. From the above discussion, we expect that there will also be logarithmic operators in any gravitationally dressed $c < 1$ model if the puncture operator appears as the dressing of one of the primary fields in the matter theory. From eqs. (41) and (40), we can see that the field that will be dressed by the puncture operator has the dimension

$$1 - h_P = -\frac{(p-q)^2}{4pq} = h_{0,0} \tag{42}$$

The puncture operator therefore appears as the dressing of the "puncture" (or pre-logarithmic) operator in the (p,q) model. We therefore expect that there will be no logarithmic operators in minimal models coupled to gravity, because there are no "puncture" (ie. no pre-logarithmic) operators, but that when logarithmic theories are coupled to gravity, there will be additional logarithmic operators in the gravity sector. The only case in which logarithmic operators exist in the gravitationally dressed model but not in the model without gravity is $c = 1$. This is because in a $c = 1$ model (without gravity), the field with the minimum dimension is the identity, but there is no "puncture" operator with the same dimension.

3 Correlation Functions

We now consider correlation functions in models which contain both operators V_α and D_α. As usual, the 2-point functions are completely determined by projective invariance,

and they can be derived either from the four-point functions as in [1, 7], or from the transformation law implied by eq. (36), as in [30] (correlation functions can also be found using the $\mathcal{W}_\infty$ algebra [31]):

$$
\begin{aligned}
& z \to z + \epsilon(z) \\
\delta D_\alpha(z) &= \partial\epsilon(z)\,[h_\alpha D_\alpha(z) + V_\alpha(z)] + \epsilon(z)\partial D_\alpha(z) \\
\delta V_\alpha(z) &= \partial\epsilon(z) h_\alpha V_\alpha(z) + \epsilon(z)\partial V_\alpha(z)
\end{aligned}
\tag{43}
$$

Correlation functions must be invariant under the projective transformations, which can be written as [30]

$$
\begin{aligned}
[L_n, V_\alpha(z)] &= z^{n+1}\partial V_\alpha + (n+1)z^n h_\alpha V_\alpha \\
[L_n, D_\alpha(z)] &= z^{n+1}\partial D_\alpha + (n+1)z^n h_\alpha D_\alpha + (n+1)z^n V_\alpha \\
& n = 0, \pm 1
\end{aligned}
\tag{44}
$$

and non-zero correlation functions must also satisfy the neutrality condition $\sum_i \alpha_i = 2\alpha_0$. There are similar relations for the $\bar{z}$ dependence. For the ordinary primary fields V_α, we find as usual:

$$
\langle V_\alpha(z,\bar{z}) V_{2\alpha_0-\alpha}(0)\rangle = \frac{A}{|z|^{4h_\alpha}}
\tag{45}
$$

In ordinary conformal field theories, the constant A is not determined, but if the logarithmic operator D_α exists, eq. (44) leads to the following equations for $\langle V_\alpha(z) D_{2\alpha_0-\alpha}(w)\rangle$:

$$
\begin{gathered}
[\partial_z + \partial_w]\,\langle V_\alpha(z) D_{2\alpha_0-\alpha}(w)\rangle = 0 \\
[z\partial_z + w\partial_w + 2h_\alpha]\,\langle V_\alpha(z) D_{2\alpha_0-\alpha}(w)\rangle + \langle V_\alpha(z) V_{2\alpha_0-\alpha}(w)\rangle = 0 \\
[z^2\partial_z + w^2\partial_w + 2h_\alpha(z+w)]\,\langle V_\alpha(z) D_{2\alpha_0-\alpha}(w)\rangle + 2w\langle V_\alpha(z) V_{2\alpha_0-\alpha}(w)\rangle = 0
\end{gathered}
\tag{46}
$$

These equations are only consistent if $A = 0$, so when the logarithmic operator D_α exists,we must have:

$$
\langle V_\alpha(z,\bar{z}) V_{2\alpha_0-\alpha}(0)\rangle = 0
\tag{47}
$$

Solving eq. (46), and the similar equations for $\langle D_\alpha(z,\bar{z}) D_{2\alpha_0-\alpha}(0)\rangle$ then leads to:

$$
\begin{aligned}
\langle V_\alpha(z,\bar{z}) D_{2\alpha_0-\alpha}(0)\rangle &= \frac{B}{|z|^{4h_\alpha}} \\
\langle D_\alpha(z,\bar{z}) D_{2\alpha_0-\alpha}(0)\rangle &= \frac{-2B\log|z|^2 + \delta}{|z|^{2h_\alpha}}.
\end{aligned}
\tag{48}
$$

Since $V_P(z)$ is an ordinary primary field, and does not have a logarithmic partner, we have:

$$\langle V_{\alpha_0}(z,\bar{z})V_P(0)\rangle = \frac{1}{|z|^{2h_{\alpha_0}}} \tag{49}$$

However, we can also compute all these two-point functions directly using eqs. (25) and (26), giving

$$\begin{aligned} \langle V_\alpha(z,\bar{z})V_{2\alpha_0-\alpha}(0)\rangle \quad &\sim \quad \frac{\langle V_{2\alpha_0}\rangle}{|z|^{2h_\alpha}} \\ \langle V_\alpha(z,\bar{z})D_{2\alpha_0-\alpha}(0)\rangle \sim \langle V_{\alpha_0}(z,\bar{z})V_P(0)\rangle \quad &\sim \quad \frac{\langle D_{2\alpha_0}\rangle}{|z|^{2h_{\alpha_0}}} \end{aligned} \tag{50}$$

Since $V_{2\alpha_0}$ is an identity operator ($h_{2\alpha_0}=0$), in ordinary CFT we would take $\langle V_{2\alpha_0}\rangle = 1$, but if any of the operators D_α or V_P exist, eqs. (47), (48), (49) and (50) will only be consistent if we have:

$$\begin{aligned} \langle V_{2\alpha_0}\rangle = \langle I\rangle \quad &= \quad 0 \\ \langle D_{2\alpha_0}\rangle \quad &= \quad 1 \end{aligned} \tag{51}$$

It can then be checked that, assuming eq. (51), we find using eq. (25) that

$$\langle D_{2\alpha_0}(z,\bar{z})D_0(0)\rangle = -2\log|z|^2 \tag{52}$$

In agreement with eq. (48).

There must therefore be at least one logarithmic operator, with dimension 0, in any model which contains the puncture operator, which is indeed the case in th $c=-2$ and other $c_{p,q}$ models [1, 4, 5, 6]. In general there will also be logarithmic operators with other dimensions, but it is necessary to actually compute the four-point functions or fusion rules to determine for which values of α the operators D_α exist. The four-point functions containing the puncture operator are given by the expressions given in [27], but as these integrals diverge they have to be analytically continued from values of c for which $\alpha_{q,p} \neq \alpha_0$, and it is in this way that the logarithmic singularities can appear. These correlation functions always contain exactly one operator $V_{2\alpha_0-\alpha_{q,p}}$, in addition to ordinary operators $V_{\alpha_{r,s}}$ and screening operators, in order to satisfy the condition $\sum\alpha = 2\alpha_0$. In the limit where $\alpha_{q,p}\to\alpha_0$, we could take either

$$V_{2\alpha_0-\alpha_{q,p}} \to V_{\alpha_0} \tag{53}$$

or

$$V_{2\alpha_0-\alpha_{q,p}} \to V_P. \tag{54}$$

The logarithms in the four- point functions tell us that eq. (54) is correct. In addition, any correlation functions containing two or more puncture operators can be seen to be the analytic continuation of functions with $\sum \alpha \neq 2\alpha_0$, and must therefore vanish. This is important as calculating these functions using eq. (25) leads to extra factors of $\log|z|$ which should not appear.

We might expect from eqs. (34) and (35) that we would obtain a logarithmic operator from the fusion of the puncture operator with any other primary operator. However, this is not always the case in the $c_{p,1}$ models. We can see why this is by observing that in some cases the OPE of the two primary fields only contains one primary field, as can be seen from eq. (8):

$$\Phi_{1,p}(z)\Phi_{s,1}(0) \sim z^{h_{s,p}-h_{1,p}-h_{s,1}}\Phi_{s,p}(0) + \text{descendants} \tag{55}$$

This means that the relevant four point function has only one conformal block. The logarithmic operators appear when two of the dimensions of fields in the OPE become degenerate – there is no way for this to happen in this case. However, as we have seen, the OPE of $\Phi_{1,p}$ with itself does contain the logarithmic operator D_0:

$$\Phi_{1,p}(z)\Phi_{1,p}(0) \sim z^{-2h_{1,p}}\left[D_0(0) + \log|z|^2 I + \cdots\right] + \ldots \tag{56}$$

This implies that logarithms must also appear in correlation functions of primary fields with dimension $h_{s,p}$, even if they do not contain the puncture operator, as can be seen by writing:

$$\begin{aligned}\Phi_{s,p}\times\Phi_{s,p} &\sim \Phi_{s,1}\times[\Phi_{1,p}\times\Phi_{1,p}]\times\Phi_{s,1} \\ &\sim \Phi_{s,1}\times\left[D_0(0)+\log|z|^2 I\right]\times\Phi_{s,1}\end{aligned} \tag{57}$$

Another way to see this is to observe that the OPE $\Phi_{s,p}(z)\Phi_{s,p}(0)$ contains two primary fields, the identity and $\Phi_{1,2p-1}$ which have the same dimension $h_{1,2p-1} = 0$ when $c = c_{p,1}$. The indicial equation for the corresponding differential equation will therefore have two equal roots, and so there will be logarithms in one of the conformal blocks. Therefore all

the $\Phi_{s,p}$ operators, not just the $\Phi_{1,p}$ operator will behave as "pre-logarithmic" operators in the $(p,1)$ model.

4 Logarithmic Operators Degenerate with Descendants

So far we have considered the logarithmic operators which occur when the dimensions of two primary fields become degenerate. for example, in the $c=-2$ model, the dimensions $h_{1,1} = h_{1,3} = 0$, giving the logarithmic pair with dimension 0. This is the situation when the roots of the indicial equation for the conformal blocks are equal. Logarithmic operators can also occur when two of the dimensions in the spectrum differ by an integer, so that one primary field becomes degenerate with a descendent of another, and this type of logarithmic operator also appear in $c_{p,q}$ models. This occurs when the roots of the indicial equation differ by an integer. In this case, we have a primary operator C^1 with dimensions $(h-n,h)$, (n is an integer), and with a null vector on the level n, and another primary operator $\bar{C}^1$ with dimensions $(h,h-n)$ (up to now, all the operators we have considered have had equal left and right dimensions). There is also a primary operator C and a logarithmic operator D with dimensions (h,h), with the relations [6]:

$$\begin{aligned} \sigma_{-n}C^1 &= C \\ \bar{\sigma}_{-n}\bar{C}^1 &= C \\ L_0 D &= hD + C \\ (L_1)^n D &= \beta C^1 \end{aligned} \tag{58}$$

Where σ_{-n} is the combination of Virasoro generators L_i which gives the null vector and β is a constant. In addition, C and C^1 satisfy the usual relations for primary operators. We can also redefine $D(z)$ by adding descendants of $C^1(z)$ so that

$$L_i D = 0, \qquad\qquad i \geq 2. \tag{59}$$

This leads to the following OPE:

$$T(z)D(0) \sim \frac{L_1 D(0)}{z^3} + \frac{hD(0)}{z^2} + \frac{C(0)}{z^2} + \frac{\partial D(0)}{z} + \cdots \tag{60}$$

As before, the operators with this behaviour have the form of derivatives with respect to α of ordinary operators. Also, since C^1 must necessarily have $h \neq \bar{h}$, we write ϕ as:

$$\phi(z,\bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z}) \tag{61}$$

The simplest example occurs for $h = 1$. In this case we have three primary fields; C with conformal weights $(1,1)$, C^1 with weights $(0,1)$ and $\bar{C}^1$ with weights $(1,0)$; and a logarithmic operator D with weights $(1,1)$, which are constructed as follows:

$$\begin{aligned}
C^1 &= \, : e^{i\alpha_\pm \bar{\varphi}(\bar{z})} : \\
\bar{C}^1 &= \, : e^{i\alpha_\pm \varphi(z)} : \\
C &= \, : e^{i\alpha_\pm \phi(z,\bar{z})} : \\
D &= \frac{d}{d\alpha}\left\{ e^{i(\alpha_\pm + \alpha)\phi(z,\bar{z}))} + \lambda L_{-1} e^{i[\alpha_\pm \bar{\varphi}(\bar{z}) + \alpha\phi(z,\bar{z})]} + \lambda \bar{L}_{-1} e^{i[\alpha_\pm \varphi(z) + \alpha\phi(z,\bar{z})]} \right\}_{\alpha=0} \\
&= i : \left[\phi(z,\bar{z}) e^{i\alpha_\pm \phi(z,\bar{z}))} + \lambda \partial\phi(z,\bar{z}) e^{i\alpha_\pm \bar{\varphi}(\bar{z})} + \lambda \bar{\partial}\phi(z,\bar{z}) e^{i\alpha_\pm \varphi(z)} \right] :
\end{aligned} \tag{62}$$

where λ has to be chosen to give the correct value of β in eq. (58). More generally, when the dimensions differ by an integer n, the logarithmic operator with the behaviour of eq. (60) can be written as:

$$D = \frac{d}{d\alpha}\left\{ : Ce^{i\alpha\phi} : + \lambda \sigma_{-n} : C^1 e^{i\alpha\phi} : \right\}_{\alpha=0} + \text{descendants of } C^1 + \text{c.c.} \tag{63}$$

5 WZNW Models for $SU(2)_0$ and $SL(2)$

Among the other models in which there are logarithmic operators are the WZNW model on the group $SU(2)$ at $k = 0$, and possibly also the $SL(2)/U(1)$ coset model. To study these models in the same way as the $c_{p,q}$ models, we use the free field representation of [32]. We introduce three free fields, u, v and ϕ, each of which obeys eq. (25). The stress tensor for the WZNW model at level k is written as:

$$T(z) = T_u(z) + T_v(z) + T_\phi(z) \tag{64}$$

where:

$$T_\varphi = -\frac{1}{4} : \partial_z \varphi \partial_z \varphi : + i\alpha_{0,\varphi} \partial_z^2 \varphi$$

$$\varphi = u,\ v,\ \phi$$

$$\alpha_{0,u}^2 = -\frac{1}{8}, \qquad \alpha_{0,u}^2 = \frac{1}{8}, \qquad \alpha_{0,\phi}^2 = \frac{1}{4(k+2)} \tag{65}$$

The currents, $J^{\pm}$ and J^0, are:

$$\begin{aligned} J^+ &= \frac{1}{\sqrt{2}}\partial v e^{(-u+iv)/\sqrt{2}} \\ J^0 &= -\frac{i\sqrt{(k+2)}}{2}\partial\phi + \frac{1}{\sqrt{2}}\partial u \\ J^- &= \frac{1}{\sqrt{2}}\left[-\sqrt{2(k+2)}\partial\phi - i(k+2)\partial u - (k+1)\partial v\right] e^{(u-iv)/\sqrt{2}} \end{aligned} \tag{66}$$

The primary fields of the WZNW model, with dimensions $h_j = \frac{j(j+1)}{k+2}$, are the vertex operators:

$$V_{j,m} = e^{-ij\phi/\sqrt{k+2}} e^{\sigma(u-iv)}, \qquad \sigma = \frac{m-j}{\sqrt{2}} \tag{67}$$

The OPEs of these operators with the currents are:

$$\begin{aligned} J^+(z)V_{j,m}(0) &= \frac{i(m-j)}{z}V_{j,m+1}(0) \\ J^0(z)V_{j,m}(0) &= \frac{m}{z}V_{j,m}(0) \\ J^-(z)V_{j,m}(0) &= \frac{i(j+m)}{z}V_{j,m-1}(0) \end{aligned} \tag{68}$$

5.1 Jordan Blocks in Affine Algebra for $SL(2)$

As in the $c_{p,q}$ models, non-zero correlation functions in the free field formulation of the WZNW model must contain one of the operators $V_{-1-j,m}$, which has the same conformal dimension as $V_{j,m}$. The equivalent of the puncture operator therefore has $j = -1 - j$, so $j = -1/2$. of course this operator, which is in an infinite dimensional representation of $SU(2)$, cannot exist in the WZNW model on the group $SU(2)$, but it could exist in the model based on the non-compact group $SL(2,\mathcal{R})$. This is obtained by simply redefining $J^{\pm}$ as $iJ^{\pm}$ in eq. (66). By analogy with the $c_{p,1}$ models, we might therefore expect the WZNW model with the non-compact group to include operators of the form

$$\tilde{V}_{j,m} = \frac{\partial}{\partial j}V_{j,m} = -\frac{i\phi}{\sqrt{k+2}} e^{-ij\phi/\sqrt{k+2}} e^{\sigma(u-iv)} \tag{69}$$

We can find the OPEs of these new operators with the currents either by differentiating eq. (68) or using eqs. (66) and (25). The result is (taking into account the extra factor of i in $J^{\pm}$):

$$
\begin{aligned}
J^{+}(z)\tilde{V}_{j,m}(0) &= \frac{(m-j)}{z}\tilde{V}_{j,m+1}(0) \\
J^{0}(z)\tilde{V}_{j,m}(0) &= \frac{m}{z}\tilde{V}_{j,m}(0) + \frac{1}{z}V_{j,m}(0) \\
J^{-}(z)\tilde{V}_{j,m}(0) &= \frac{(j+m)}{z}\tilde{V}_{j,m-1}(0) + \frac{2}{z}V_{j,m-1}(0)
\end{aligned} \tag{70}
$$

From eqs. (68) and (70), we can see that, just as the logarithmic operators D_α together with the ordinary primary operators formed Jordan blocks for L_0, the V and $\tilde{V}$ operators form Jordan blocks for the zero-modes of the currents:

$$
\begin{aligned}
J_0^0 V_{j,m} &= mV_{j,m} & J_0^0 \tilde{V}_{j,m} &= m\tilde{V}_{j,m} + V_{j,m} \\
J_0^+ V_{j,m} &= (m-j)V_{j,m+1} & J_0^+ \tilde{V}_{j,m} &= (m-j)\tilde{V}_{j,m+1} \\
J_0^- V_{j,m} &= (m+j)V_{j,m-1} & J_0^- \tilde{V}_{j,m} &= (m+j)\tilde{V}_{j,m-1} + 2V_{j,m-1}
\end{aligned} \tag{71}
$$

However, it turns out that the $\tilde{V}$'s are not logarithmic operators, at least in the WZNW model, as we can see by calculating the OPE with the stress tensor, which has the Sugarawa form:

$$
T(z) = \frac{1}{k+2} : J^a(z)J^a(z) := \frac{1}{k+2} : \left(J^0J^0 - \frac{1}{2}J^+J^- - \frac{1}{2}J^-J^+\right) : \tag{72}
$$

When we calculate $T(z)\tilde{V}_{j,m}(0)$ using eq. (70), the $V_{j,m}$ terms all cancel, leaving the OPE for a primary field of the Virasoro algebra:

$$
T(z)\tilde{V}_{j,m}(0) = \frac{j(j+1)}{k+2}\frac{\tilde{V}_{j,m}(0)}{z^2} + O\left(\frac{1}{z}\right) \tag{73}
$$

In fact, we can see that this had to be true, because $L_0 = J_0^a J_0^a/(k+2)$ is just the Casimir operator for $SL(2)$ and must therefore be diagonalizable, so there can be no non-trivial Jordan blocks.

However, the $\tilde{V}$'s may become logarithmic operators in models with modified stress tensors, in which L_0 is not a Casimir operator. There are two examples of such models in which logarithmic operators do exist [23]. One is 2 dimensional gravity, for which the

stress tensor can be written as [33]:

$$T(z) = \frac{1}{k+2} : J^a(z)J^a(z) : + \frac{\partial}{\partial z} J^0(z) \tag{74}$$

If we compute the OPE of $\tilde{V}_{j,m}$ with this stress tensor, the first term, which is just the Sugarawa form, again only gives $\tilde{V}_{j,m}$, but the second term gives us the mixing with $V_{j,m}$ which characterizes logarithmic operators:

$$T(z)\tilde{V}_{j,m}(0) = \left(\frac{j(j+1)}{k+2} - m \right) \frac{\tilde{V}_{j,m}(0)}{z^2} - m \frac{V_{j,m}(0)}{z^2} + O\left(\frac{1}{z}\right) \tag{75}$$

This has the form of the OPE for a logarithmic operator D (eq. (21)), with $D = \tilde{V}_{j,m}$ and $C = -mV_{j,m}$. The logarithmic operators found in [3] are examples of this type of operator.

The second model in which this happens is the $SL(2)/U(1)$ coset model, which describes the Witten 2D black hole [34], with the stress tensor:

$$T(z) = \frac{1}{k+2} : J^a(z)J^a(z) : - \frac{1}{k} : J^0(z)J^0(z) : \tag{76}$$

As before, the second term leads to mixing between $\tilde{V}_{j,m}$ and $V_{j,m}$:

$$T(z)\tilde{V}_{j,m}(0) = \left(\frac{j(j+1)}{k+2} - \frac{m^2}{k} \right) \frac{\tilde{V}_{j,m}(0)}{z^2} - \frac{2m}{k} \frac{V_{j,m}(0)}{z^2} + O\left(\frac{1}{z}\right) \tag{77}$$

Again, this has the form of eq. (21) with $D = k\tilde{V}_{j,m}$ and $C = -2mV_{j,m}$. It is therefore possible for logarithmic operators to exist in a coset model even if they did not exist in the original WZNW model.

5.2 Logarithmic Operators in the WZNW Model for $SU(2)$ at $k = 0$

From the above discussion, we can see that the only way it is possible for logarithmic operators to appear in a WZNW model is if they are not annihilated by all positive modes of the currents. For example, if

$$J_1^a D \neq 0 \tag{78}$$

then L_0 will not be a Casimir operator:

$$L_0 = \frac{1}{k+2} : J_n^a J_{-n}^a := \frac{1}{k+2}\left[J_0^a J_0^a + 2J_{-1}^a J_1^a\right] \tag{79}$$

We will therefore have a representation of the algebra of the type (3) instead of (2). These logarithmic operators will be of the type discussed in the previous section with

$$L_1 D = \frac{2}{k+2} J_0^a J_1^a D \neq 0 \tag{80}$$

One model in which this happens is the WZNW model for $SU(2)$ at $k = 0$. To see why logarithmic operators should be expected in this model, we concentrate on the ϕ dependent parts of the stress tensor $T(z)$ and primary operators $V_{j,m}$, since the u and v dependent parts are independent of k. The ϕ dependent part of the stress tensor $T(z)$ is in fact identical to the stress tensor for the $c = -2$ model, as can be seen by comparing eqs. (23) and (24) with:

$$T_\phi(z) = -\frac{1}{4} : \partial_z\phi\partial_z\phi : -\frac{i}{\sqrt{8}}\partial_z^2\phi. \tag{81}$$

The primary fields in the two models (without the u and v dependent parts) are also the same: comparing eqs. (28) and (67), we see that

$$\begin{array}{lcll} V_j(k=0) & \sim & V_{j+1,1}(c=-2) & j = 0, 1, \ldots \\ V_j(k=0) & \sim & V_{j+\frac{3}{2},2}(c=-2) & j = \frac{1}{2}, \frac{3}{2}, \ldots \end{array} \tag{82}$$

where $V_{r,s}(c = -2)$ is the operator with dimension $h_{r,s}$ in the $c = -2$ model, and the dimensions h_j and $h_{r,s}$ are equal. The puncture operator with dimension $h_{1,2} = -\frac{1}{8}$ therefore corresponds to $j = -\frac{1}{2}$ and does not exist in the WZNW model for $SU(2)$, but, as discussed earlier, logarithmic operators also appear in the OPE of any of the fields with dimension $h_{r,2}$ in the $c = -2$ model, and so we expect them to appear also in the OPE of any of the fields with half-integer j in the $k = 0$ model. The case of $j = 1/2$ was studied in [10], where the following OPE was found:

$$\begin{aligned} g_{\epsilon_1\bar{\epsilon}_1}(z_1,\bar{z}_1) g^\dagger_{\bar{\epsilon}_2\epsilon_2}(z_2,\bar{z}_2) = |z_{12}|^{-3/2} \times & \Big\{ z_{12}\delta_{\bar{\epsilon}_1\bar{\epsilon}_2} t^i_{\epsilon_1\epsilon_2} K^i(z_2) + \bar{z}_{12}\delta_{\epsilon_1\epsilon_2}\bar{t}^i_{\bar{\epsilon}_1\bar{\epsilon}_2}\bar{K}^i(\bar{z}_2) \\ & + |z_{12}|^2 t^i_{\epsilon_1\epsilon_2}\bar{t}^j_{\bar{\epsilon}_1\bar{\epsilon}_2}\left[D^{ij}(z_2,\bar{z}_2) + \ln|z_{12}| C^{ij}(z_2,\bar{z}_2)\right] + \cdots \Big\} \end{aligned} \tag{83}$$

Here K and $\bar{K}$ are primary fields with dimensions $(1,0)$ and $(0,1)$, and C and D are the logarithmic pair with dimensions $(1,1)$. From the above discussion we expect to find $J_1^a D \neq 0$, and we can check this using the OPE for the currents with the primary field g:

$$J^a(w) g_{\epsilon_1 \bar{\epsilon}_1}(z_1) g^{\dagger}_{\bar{\epsilon}_2 \epsilon_2}(z_2) = \frac{1}{w - z_1} t^a_{\epsilon_1 \eta_1} g_{\eta_1 \bar{\epsilon}_1}(z_1) g^{\dagger}_{\bar{\epsilon}_2 \epsilon_2}(z_2) + \frac{1}{w - z_2} g_{\epsilon_1 \bar{\epsilon}_1}(z_1) g^{\dagger}_{\bar{\epsilon}_2 \eta_2}(z_2) t^a_{\eta_2 \epsilon_2} + \cdots \tag{84}$$

Taking the limit as $z_1 \to z_2$ in eq. (84) using eq. (83), we find:

$$J^a(w) t^i_{\epsilon_1 \epsilon_2} D^{ij}(z, \bar{z}) = \frac{t^a_{\epsilon_1 \eta_1} t^i_{\eta_1 \epsilon_2}}{(w - z)^2} \bar{K}^j(\bar{z}) + \frac{f^{aib}}{w - z} t^b_{\epsilon_1 \epsilon_2} D^{bj}(z, \bar{z}) + \cdots \tag{85}$$

from which we can see that, as expected, $J_1^a D \sim \bar{K}$.

As was pointed out in [10], K and $\bar{K}$ are conserved currents, indicating that the WZNW model at $k = 0$ has an additional symmetry (as well as the $SU(2)$ symmetry), which we can also try to understand in terms of the relation with the $c = -2$ model. The extended symmetry in the $c = -2$ model has a $\mathcal{W}$ algebra, which is generated by the dimension 3 fields, $\Phi_{3,1}$ [35]. There are dimension 3 fields in the $k = 0$ model as well (the primary fields with $j = 2$), and we might conjecture that these also generate a $\mathcal{W}$ algebra. However, the operator product expansions, and therefore the algebra, of the $j = 2$ fields in the $k = 0$ model and the $\Phi_{3,1}$ fields in the $c = -2$ model are not the same. In the $c = -2$ model, we have:

$$\Phi_{3,1} \times \Phi_{3,1} \sim [\Phi_{1,1}] + [\Phi_{3,1}] + [\Phi_{5,1}] \tag{86}$$

while in the WZNW model, we have:

$$V_2 \times V_2 \sim [V_0] + [V_1] + [V_2] + [V_3] + [V_4] \tag{87}$$

The appearance of V_1 in eq. (87), when the corresponding operator $\Phi_{2,1}$ does not appear in eq. (86), will lead to extra terms in the singular part of the OPE for $V_2(z)V_2(0)$, so that the algebra of these operators is not the same as the algebra of the $\Phi_{3,1}$ operators in the $c = -2$ model. The reason why the correspondence between operators in the two models does not extend to the OPEs (or correlation functions) is that the screening operators $Q = \oint J$ used to construct correlation functions are different. In the $c = -2$ model, $J = e^{i\alpha_\pm \phi}$, where $\alpha_\pm$ is given by eq. (29), while in the $k = 0$ model we have a completely

different expression $J = -ie^{(-i\phi-u+iv)/\sqrt{2}}\partial v$. It remains a difficult problem to determine the full symmetry algebra of the $k = 0$ model.

6 Conclusion

In this paper, we have shown that the logarithmic operators which were known to exist in several different conformal field theories – $c_{p,1}$ models, gravitationally dressed CFT and some WZNW and coset models – can be understood in the free field formulation of the models as originating from a "pre-logarithmic" operator, the puncture operator. The logarithmic operators are derivatives of the ordinary primary vertex operators, while the puncture operator is the only ordinary primary operator that has the form of a logarithmic operator. The existence of the puncture operator leads naturally to the existence of logarithmic operators, and this gives us an easy way to guess when logarithmic operators will appear in other models, where it may be difficult or impossible to explicitly compute the conformal blocks.

We have also shown that in some models there may be operators which form Jordan blocks for the Kac-Moody algebra, in the same way as the logarithmic operators form Jordan blocks for the Virasoro algebra. These occur in physically interesting models, such as gravitationally dressed CFT and the model of 2D black holes. The further investigation of this subject is of great interest.

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